ON THE CONFORMAL REPRESENTATION OF ALEXANDROV SURFACES

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Abstract.

We consider two-dimensional manifolds of bounded curvature. The theory of two-dimensional manifolds of bounded curvature can be viewed as general Alexandrov’s surface theory. A metric space \((M, \rho)\) is called a two-dimensional manifold of bounded curvature if the following conditions are satisfied:

1. \((M, \rho)\) is homeomorphic to a two-dimensional manifold;
2. The metric \(\rho\) is intrinsic: for every pair of points \(X, Y \in M\), the distance \(\rho(X, Y)\) is the greatest lower bound of lengths of curves joining \(X\) to \(Y\);
3. For every point \(P \in M\), there is a sequence of Riemannian metrics \((\rho_\nu)_{\nu \in \mathbb{N}}\) which are defined in a neighborhood \(U\) of the point \(P\) and such that the functions \(\rho_\nu\) converge uniformly to \(\rho_U = \rho|_U\) on the set \(U \times U\) and the sequence \(\left(|\omega_\nu|(U)\right)_{\nu \in \mathbb{N}}\) is bounded, where

\[
|\omega_\nu|(U) = \int\int_U K_\nu dS_\nu,
\]

\(K_\nu\) is the Gaussian curvature of the Riemannian metric \(\rho_\nu\) and \(dS_\nu\) is the area element.

The basic concepts of the smooth Gaussian theory of surfaces can be generalized to non-regular Alexandrov’s surfaces; however, pointwise metric invariants of a surface should be substituted with certain integral invariants for general Alexandrov’s surfaces. Thus, the pointwise Gaussian curvature is replaced with an additive set function which, in a smooth case, coincides with the integral curvature of a set. In general case, this generalized integral curvature is not absolutely continuous. Examples of two-dimensional spaces of bounded integral curvature include:

1. An arbitrary convex surface in \(\mathbb{R}^3\);
2. An arbitrary polyhedral surface in \(\mathbb{R}^3\);
3. A surface that is formed by tangents to an arbitrary regular curve, base curve, in \(\mathbb{R}^3\).
For polyhedral metric, the integral curvature is a discrete set function concentrated on the set all points of which are isolated. In case 3, the integral curvature is concentrated on the base curve.

The main result states

Let $M$ be a two-dimensional manifold of bounded curvature. Then every point $X \in M$ has a neighborhood $U$ such that the metric of the manifold in this neighborhood can be given by a linear element of the form

$$ds^2 = \lambda(x, y) (dx^2 + dy^2)$$

where the function $\log \lambda(x, y)$ is the difference of two subharmonic functions. The converse is also true: if, locally, the metric $\rho$ on a two-dimensional manifold $M$ is given by $ds^2 = \lambda(x, y) (dx^2 + dy^2)$, and $\log \lambda(x, y)$ is the difference of two subharmonic functions, then $M$ is a two-dimensional manifold of bounded curvature.

The coordinates in which the linear element is given by $ds^2 = \lambda (dx^2 + dy^2)$ are called isothermal. We intend to give some applications of general isothermal coordinates in geometry and complex analysis.