Generalizations of Stirling’s formula

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Stirling’s formula can be regarded as a method of deducing the asymptotic behavior of the coefficients of the exponential function from that of the function on the positive real axis.

It turns out that the method can be extended to a wide class of functions called “admissible”. If

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is such a function, we write

$$a(r) = r f'(f)/f(r), \quad b(r) = r a'(r),$$

and define $r_n$ by $a(r_n) = n$. Then

$$a_n \sim r_n^{-n} f(r_n) (2\pi b(r_n))^{-1/2}, \quad n \to \infty.$$  

There are a number of applications, particularly in Group Theory, when $a_n$ represents the number of a certain class of objects. The case when $f(z) = \exp P(z)$ has been considered by a number of authors including Wilf [1986], Müller [1997] and Liebeck and Shalev [to be published]. The general theory and some of these consequences will be described.