Discrete integrability and cluster algebras

Classical Liouville integrability has a discrete analogue, with Poisson-commuting integrals of motion with respect to a discrete time evolution. The canonical quantization is a discrete quantum integrable system with commuting, time-independent Hamiltonians. Cluster algebras, introduced by Fomin and Zelevinsky, are a rich source of discrete evolutions with a canonical Poisson structure compatible with the evolution, as described by Gekhtman, Shapiro and Vainshtein. In some cases they are integrable evolutions. In addition to a beginner’s guide to cluster algebras and their quantization, I will introduce two integrable examples from the representation theory of quantum groups, T-systems and Q-systems. The story will culminate with the (possibly surprising) appearance of discrete quantum Toda Hamiltonians and family of generalizations, acting on the characters of generalized Weyl modules or fusion products. The entire construction has a deformation beyond quantization, into Macdonald theory, elliptic Hall algebras, spherical DAHAs and other wonderful objects from quantum field theory.