ON NEGATIVITY OF HIGHER EULER CHARACTERISTICS

S. P. Dutta

Abstract. Positivity of higher Euler characteristics on equicharacteristic and unramified regular local rings follow from the works of Serre, Lichtenbaum, and Hochster. Here we prove that such higher Euler characteristics could be negative on complete intersections even for a pair of intersecting modules, both having finite projective dimension, satisfying Serre-vanishing.

Throughout this work $R$ denotes a Noetherian local ring and $M$, $M'$, $N$, $N'$, etc. denote finitely generated $R$-modules. In particular, suppose that $M$ and $N$ are modules such that $\ell(M \otimes_R N) < \infty$ (here $\ell$ stands for “length”) and $M$ has finite projective dimension. With this hypothesis, one can define $\chi_i^R(M, N)$ as $\sum_{j \geq 0} (-1)^{j} \ell(\text{Tor}^{R}_{i+j}(M, N))$ for every $i \geq 0$. Here our main focus is on the behaviour of such functions when both $M$ and $N$ have finite projective dimension and $i > 0$.

Serre proved [S] that $\chi_i^R(M, N) \geq 0$ when $R$ is any equicharacteristic regular local ring. Next Lichtenbaum [Li] proved the same result for unramified regular local rings (see Background for details). In this paper we would like to prove that $\chi_i(i \geq 2)$, specifically $\chi_2$, can be negative over complete intersections and Gorenstein Rings even when vanishing holds (i.e. $\chi_0^R(M, N) = 0$) in the above set-up (i.e. $pd_R M < \infty$ and $pd_R N < \infty$). This is very surprising because vanishing does hold over complete intersections when both

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modules have finite projective dimension, due to a theorem of P. Roberts [R], and the usual expectation is that if vanishing holds, the other intersection multiplicity related conjectures of Serre (see Background) should also be valid.

In [D-H-M] Hochster, Mclaughlin and this author constructed modules $M$ of finite length and finite projective dimension over a 3-dimensional local hypersurface $R = k[x, y, u, v]/(xy - uv)$, $m$ being the maximal ideal generated by $x, y, u$ and $v$) with negative intersection multiplicity. In this case $\chi_0(M, R/P) = -1$ where $P = (x, u)$; thus vanishing fails. Afterwards Levine [L] extended it to higher dimensions (e.g $R = k[x_1, \ldots, x_n, y_1, \ldots, y_n]/(\Sigma x_i y_i)$) by using $K$-theoretic techniques. Recently Roberts and Srinivas [R-S] proved the existence of many such examples of hypersurfaces (smooth cubic-surfaces in $\mathbb{P}^3$) and Gorenstein rings (co-ordinate rings of $\mathbb{P}^n \times \mathbb{P}^m$) by using local Chern characters and the localization exact sequence from $K$-theory of Thomason and Trobaugh. We will show that all the hypersurfaces, mentioned here, possess counterexamples to the $\chi_i$-conjecture.

One of the main ideas in our proof stems from one of our previous results (theorem 3.2, [D1]) where we showed that positivity (when one of the modules is perfect) implies vanishing over Gorenstein rings. Two related Theorems—one due to Auslander and Bridger and the other due to Auslander and Buchweitz are used in our work. In (1.3) we sketch a short and direct proof for both. Next we assert that for positivity of $\chi(M, N)$, $M$ perfect, one can assume depth $N = \dim N - 1$ (Prop. (1.4)). This leads us to introduce the notion of a companion module. Given a pair of intersecting modules $M, N$ (i.e. $\ell(M \otimes_R N) < \infty$) we say $N'$ is a companion module of $N$ if (i) $\dim N' = \dim N$, (ii) depth $N' = \dim N' - 1$ and (iii) $\chi(M, N) = \chi(M, N')$. This notion plays a crucial role in the proof of our main
Theorem (1.9). Another important role is played by Theorem (1.6) where we deduce a sufficient condition for non-negativity.

**Statement of the Main Theorem.**

**Theorem (1.9).** Let $R$ be a Gorenstein local ring for which vanishing fails. We have the following:

1. There exists a perfect module $M_r$ and a prime ideal $P$ of $R$ such that $\ell(M_r \otimes R/P) < \infty$, $\dim M_r + \dim R/P = \dim R - 1$, $\chi(M_r, R/P) > 0$ and for every $R$-module $Q$ with $\dim Q < \dim R/P$, $\chi(M_r, Q) = 0$.

2. With $P$, $M_r$ as in part (1) and $\{y_1, \ldots, y_{n-d}\}$ a maximal $M_r$ sequence contained in $P$, we let $\tilde{R} = R/(y_1, \ldots, y_{n-d})$, $\tilde{M}_r = M_r/(y_1, \ldots, y_{n-d})M_r$ and $\overline{P} = P/(y_1, \ldots, y_{n-d})$. If for any such sequence $\{y_1, \ldots, y_{n-d}\}$, the ideal $\overline{P}^d$ of $\tilde{R}$ has a companion module $N$ over $\tilde{R}$ such that $\chi^R(\tilde{M}_r, \text{Ext}_1^R(N, \tilde{R})) = \chi(M_r, \text{Ext}_1^{n-d+1}(N, R)) \geq 0$, then there exists a module $N'$ of finite projective dimension over $\tilde{R}$ (hence over $R$) such that $\ell(M_r \otimes N') < \infty$, $\dim M_r + \dim N' < \dim R$, $\chi(\tilde{M}_r, N') = 0$ and $\chi_2(\tilde{M}_r, N') < 0$. Here $\tilde{M}_r = \text{Ext}^{\dim R - \dim M_r}(M_r, R)$.

Henceforth, the condition mentioned in part (2) of the above theorem will be denoted by ($\ast$). This condition may appear to be a little strict. However, by part (1) in the above theorem, the fact that $\chi(M_r, Q) = 0$, $\forall$ $R$-module $Q$ whenever $\dim Q < \dim R/P$, makes ($\ast$) valid whenever $N$ is torsionless over $\tilde{R}$. And exactly this is what happens in all the examples of the hypersurfaces mentioned above (Corollary to Theorem (1.9)). Moreover, $\chi_2$ is also negative on any three-dimensional Gorenstein local ring (Theorem 1.10), where vanishing fails. We suspect that whenever a counterexample to vanishing can be
described in somewhat concrete terms on a Gorenstein ring $R$, the condition ($\ast$) will be easily varifiable.

In (2.1) we point out the following relation between $\chi$ and $\chi_2$ over ramified regular local rings: the validity of the $\chi_2$-conjecture (see Background) implies positivity when one of the intersecting modules is Cohen-Macaulay.

**Background.** Let me present a brief background of the whole set of questions in this connection.

Let $R$ be a local ring and $M$ and $N$ be two finitely generated modules over $R$ such that one of them has finite projective dimension. We assume that $\ell (M \otimes_R N) < \infty$. Then, following Serre [S], we define

$$
\chi_i^R (M, N) = \sum_{j \geq 0} (-1)^j \ell (\text{Tor}_{i+j}^R (M, N)).
$$

When $i = 0$, we replace $\chi_0^R$ by $\chi^R$ and in addition we drop the symbol “$R$” from this notation and from Tor, Ext etc. when there is no chance of ambiguity. In [S], Serre made the following conjectures:

**Conjecture 1.** Let $R$ be a regular local ring. Then $\chi (M, N) \geq 0$; the sign of equality holds if and only if $\dim M + \dim N < \dim R$.

When $\dim M + \dim N < \dim R$ we use the term *vanishing* conjecture (or simply *vanishing*) and when $\dim M + \dim N = \dim R$ we denote it by the *positivity* conjecture (or simply *positivity*). This conjecture makes sense over any local ring when one of the modules has finite projective dimension and we extend the conjecture to this broader context.
Conjecture 2. Let R be regular local. Then $\chi_i(M, N) \geq 0$; the sign of equality holds if and only if $\text{Tor}^R_{i+j}(M, N) = 0$, $\forall j \geq 0$.

This conjecture also makes sense when $R$ is any local ring and when one of the modules has finite projective dimension. Here we adopt this broader context.

Serre [S] proved Conjecture 1 when the regular local ring $R$ is equicharacteristic or unramified. Roberts [R] and Gillet and Soule [G-S] independently proved the vanishing part; Roberts proof extends to the case when $R$ is a local complete intersection and both modules have finite projective dimension. The same was also established over Gorenstein rings up to dimension 5 in [D4]. In the mid-nineties Gabber [B] proved the non-negativity part of this conjecture.

Serre [S] proved Conjecture 2 for equicharacteristic regular local rings. Next Lichtenbaum [Li] proved that over unramified regular local rings this conjecture holds for $i \geq 2$; he also showed that it is true for $i = 1$ if one of the modules is torsion-free. Later Hochster [H2] proved the case $i = 1$ completely.

Conjecture 2 and the positivity part of Conjecture 1 are both very much open over ramified regular local rings. The invalidity of the generalized conjectures, when only one of the intersecting modules is of finite projective dimension, has already been mentioned. In this connection, I would also like to draw the reader’s attention to Heitmann’s counterexample [He] to the “Rigidity of Tor” conjecture.

One final comment: Both Serre and Lichtenbaum used the method of diagonalization to establish their result. Apparently this does not work in the non-smooth case.

In this article all rings are noetherian local and all modules are finitely generated. For
a module $M$ of finite length, we write $\hat{M}$ to denote its Matlis dual. We call a module $M$
perfect over a local ring $R$ if $M$ is Cohen-Macaulay and the projective dimension of $M$
over $R$ is finite. In this note $\text{pd}_R M$ stands for projective dimension of $M$ over $R$, $\dim M$
denotes the Krull dimension of the module $M$, $M^*$ stands for $\text{Hom}_R(M,R)$ and for any
perfect module $T$, $\overline{T}$ denotes $\text{Ext}_R^{\dim T}(T,R)$.

**Section 1**

We will refer the reader to several theorems and lemmas from [A-B], [A-Bu], [D1] and
[D2]. Proofs of a few of these results will be sketched here for the purpose of completeness
of this paper.

1.1. First we state a lemma due to Lichtenbaum [Li] (without any proof) which will be
used several times in this work.

**Lemma 1.** (Lemma 1, [Li]) Let $(R, m)$ be a local ring and let $x_1, \ldots, x_d$ be an $R$-sequence
contained in $m$. Let $\underline{x}$ denote the ideal $(x_1, \ldots, x_d)$. Let $M$ be a finitely generated $R$-module
such that $\ell(M/\underline{x}M) < \infty$. Then $\chi(R/\underline{x}, M) \geq 0$, with the equality holding if and only if
$\dim M < d$.

Next we prove the following lemma.

**Lemma 2.** Let $R$ be Cohen-Macaulay. The vanishing conjecture holds if and only if for
any perfect module $M$ and any Cohen-Macaulay module $N$ such that $\ell(M \otimes_R N) < \infty$ and
$\dim M + \dim N = \dim R - 1$, $\chi(M, N) = 0$.

*Proof.* Hochster [H1] already established the above fact over regular local rings. Similar
arguments work here too. The main point is the following: Given any pair of modules $M_1$, $N_1$ with $\text{pd} M_1 < \infty$, $\dim M_1 + \dim N_1 < \dim R$, one can always choose exact sequences

$$0 \to T_r \to (R/\mathfrak{x})^{\ell_r} \to T_{r-1} \to 0, \ldots, 0 \to T_1 \to (R/\mathfrak{x})^{\ell_1} \to N_1 \to 0$$

such that $\dim N_1 = \dim R/\mathfrak{x}$, $\ell(M_1/\mathfrak{x}M_1) < \infty$ and $T_r$ is Cohen-Macaulay with $\dim T_r = \dim N_1$. Here $\mathfrak{x}$ is the ideal generated by $x_1, \ldots, x_s$ which form a maximal $R$-sequence contained in $\text{ann}_R N_1$. By Lemma 1 (1.1), we have $\chi(M_1, R/\mathfrak{x}) = 0$. Then $\chi(M_1, N_1) = \pm \chi(M_1, T_r)$. We repeat this process for $M_1$ and thus can assume both $M_1$ and $N_1$ are Cohen-Macaulay, $\dim M_1 + \dim N_1 < \dim R$ and $M_1$ is perfect. Using the fact that $ht \text{ann}_R M_1 + ht \text{ann}_R N_1 > \dim R$ i.e. $ht \text{ann}_R M_1 > \dim N_1$, we choose $R$-sequence \( \{y_1, \ldots, y_t\} \) in $\text{ann}_R M_1$ in such a way $\ell(N_1/y_1 N_1) < \infty$ and $\dim(R/y_1) + \dim N_1 = \dim R - 1$. Now repeating the process described above, we construct the new $M$ such that $M$ is perfect, $\dim M + \dim N_1 = \dim R - 1$, $\ell(M \otimes_R N_1) < \infty$ and $\chi(M, N_1) = \pm \chi(M_1, N_1)$. Hence the lemma follows.

1.2 Theorem. Let $R$ be a Gorenstein ring. Let $M$ be perfect and let $N$ be Cohen-Macaulay such that $\ell(M \otimes_R N) < \infty$ and $\dim M + \dim N \leq \dim R$. Let $i = \dim R - \dim M - \dim N$. Then $\chi(M, N) = (-1)^i \chi(\bar{M}, \bar{N})$ where $\bar{M} = \text{Ext}^r(M, R)$, $\bar{N} = \text{Ext}^s(N, R)$, $r = \dim R - \dim M$ and $s = \dim R - \dim N$.

For a proof we refer the reader to Theorem 2.2 in [D1].

1.3. Here we will state two theorems which will play crucial roles in the proofs of our theorems. We would like to sketch a short and direct proof which works for both simultaneously.
Theorem 1. (Auslander-Bridger, Th (4.6) in [A-B]) Let $M$ be a module over a Gorenstein ring $R$. Then there exists a short exact sequence

$$0 \to L \to M \oplus R^t \to Q \to 0,$$

where $Q$ is a module with $\text{pd}_R Q < \infty$ and $L$ is a maximal Cohen-Macaulay module.

Theorem 2. (Auslander-Buchweitz, Th (1.8) in [A-Bu]) Let $M$ be a module over a Gorenstein ring $R$. Then there exists a short exact sequence

$$0 \to M \to Q \to L \to 0$$

such that $\text{pd}_R Q < \infty$ and $L$ is maximal Cohen-Macaulay over $R$.

Proof. Suppose that $\text{Ext}_R^i(M, R) = 0$ for $i > n$. Consider a minimal free resolution $F_\bullet$ of $M$ over $R$:

$$F_\bullet : \rightarrow R^{t_{n+1}} \rightarrow R^{t_n} \xrightarrow{d_n} R^{t_{n-1}} \rightarrow \cdots \rightarrow R^{t_1} \xrightarrow{d_1} R^{t_0} \rightarrow 0.$$

Apply $\text{Hom}_R(F_\bullet, R)$ and obtain $F_\bullet^{*\dagger}$ (t for truncated!)

$$F_\bullet^{*\dagger} : 0 \rightarrow R^{t_0} \xrightarrow{d_0^*} R^{t_1} \rightarrow \cdots \rightarrow R^{t_{n-1}} \xrightarrow{d_n^*} R^{t_n} \rightarrow 0$$

write $G = \text{Coker} d_n^*$. Note that we have a short exact sequence

$$0 \to \text{Ext}_R^n(M, R) \to G \to \text{Im} d_{n+1}^* \to 0$$

where $\text{Im} d_{n+1}^*$ is maximal Cohen-Macaulay. Then by Proposition 1.1 in [D3] we can obtain a free complex $L_\bullet : \rightarrow R^{b_n} \rightarrow R^{b_{n-1}} \rightarrow \cdots \rightarrow R^{b_0} \rightarrow 0$ with $H_0(L_\bullet) = \text{Ext}_R^n(M, R)$ and a
map \( \phi_\bullet : L_\bullet \to F_\bullet^{\ast t} \) such that the mapping cone of \( \phi_\bullet \) is a free resolution of \( \text{Im} \ d_{n+1}^* \). We have the following diagram

\[
\begin{array}{cccccccccccc}
R^{b_{n+1}} & \longrightarrow & R^{b_n} & \longrightarrow & R^{b_{n-1}} & \longrightarrow & \cdots & \longrightarrow & R^{b_0} & \longrightarrow & 0 \\
\downarrow \phi_n & & \downarrow \phi_{n-1} & & & & & & \downarrow \phi_0 & & \\
0 & \longrightarrow & R^{t_0} & \longrightarrow & R^{t_1} & \longrightarrow & \cdots & \longrightarrow & R^{t_n} & \longrightarrow & 0
\end{array}
\]

\((\ast)\)

Apply \( \text{Hom}(\cdot, R) \) to \((\ast)\) and we have a commutative diagram

\[
\begin{array}{cccccccccccc}
\longrightarrow & R^{t_n} & \longrightarrow & R^{t_{n-1}} & \longrightarrow & \cdots & \longrightarrow & R^{t_1} & \longrightarrow & R^{t_0} & \longrightarrow & M & \longrightarrow & 0 \\
\phi_0^* & \bigg| & \phi_1^* & \bigg| & \cdots & \bigg| & \phi_n^* & \bigg| & i & \bigg| & \phi_{n+1}^* & \bigg| & \bigg| & i & \bigg| & 0 \\
0 & \longrightarrow & R^{b_0} & \longrightarrow & R^{b_1} & \longrightarrow & \cdots & \longrightarrow & R^{b_{n-1}} & \longrightarrow & R^{b_n} & \longrightarrow & Q & \longrightarrow & 0
\end{array}
\]

\((\ast\ast)\)

where \( Q = \text{Coker} \alpha_n^* \). It is easy to check that the horizontal row is a minimal free resolution of \( Q \), \( M \xrightarrow{i} Q \) is injective and \( \text{Coker} \ i \) is maximal Cohen-Macaulay (Since the mapping cone of \( \phi \) is a free resolution of \( \text{Im} \ d_{n+1}^* \), which is maximal Cohen-Macaulay). Moreover, we have a short exact sequence

\[
0 \rightarrow L \rightarrow M \oplus R^{b_n^*} \rightarrow Q \rightarrow 0
\]

Since the mapping cone of \( \phi_\bullet^* \) is a free resolution of \( \text{coker} \ i \), we have \( \text{Ext}^i(Q, R) \simeq \text{Ext}^i(M, R) \) for \( i > 0 \). Hence \( L \) is maximal Cohen-Macaulay.

Thus both Theorems are proved.

1.4. Let us now state a simple Lemma. The proof is left to the reader.

**Lemma.** Let \( R \) be a local ring and let \( M \) and \( N \) be two finitely generated \( R \)-modules such that \( \ell(M \otimes N) < \infty \). Suppose that the annihilator of \( N \) contains an \( R \)-sequence \( x_1, \ldots, x_r \) such that it is also an \( M \) sequence. Write \( \bar{R} = R/(x_1, \ldots, x_r) \) and \( \bar{M} = M/(x_1, \ldots, x_r)M \). Then \( \chi_i^R(M, N) = \chi_i^R(\bar{M}, N) \) for \( i \geq 0 \).
**Proposition.** Let $R$ be a Gorenstein ring and let $M$ be a perfect module. In order that $M$ satisfies the positivity conjecture i.e. $\chi(M, N) > 0$ whenever $\ell(M \otimes N) < \infty$ and $\dim M + \dim N = \dim R$, it is enough to prove this when depth $N = \dim N - 1$.

**Proof.** Let $\dim M = r$, $\dim N = s$ and $I = \text{ann}_R N$. Since $\ell(M \otimes N) < \infty$ and $r + s = n$, it follows that $I$ contains an $R$-sequence $\{x_1, \ldots, x_r\}$ which is also a maximal $M$-sequence.

Let $\mathfrak{a}$ denote the ideal $(x_1, \ldots, x_r)$. We write $\tilde{R} = R/\mathfrak{a}$ and $\tilde{M} = M/\mathfrak{a}M$. If $\text{pd}_R N < \infty$, we have an exact sequence $0 \to \tilde{R}^d \to N \to H \to 0$. Here $\dim H < \dim N$, $\tilde{R} = R/\mathfrak{a}$, $\mathfrak{a}$ being the ideal generated by $\{x_1, \ldots, x_r\}$ and $d = \text{rank}_R N$. Then, by the above Lemma, $\chi^R(M, N) = \chi^R(\tilde{M}, N) = d\ell(\tilde{M})$ (since $\ell(\tilde{M}) < \infty$, $\chi^R(\tilde{M}, H) = 0$). So we may assume $\text{pd}_R N = \infty$. By Theorem 1 (1.3), we obtain an exact sequence

$$0 \to L \to \tilde{R}^t \oplus N \to Q \to 0 \cdots \quad (1)$$

where $\text{pd}_R Q < \infty$ and $L$ is maximal Cohen-Macaulay over $\tilde{R}$. If $\dim Q = \dim \tilde{R}$, let $d' = \text{rank}_R Q$. Then we obtain an exact sequence $0 \to \tilde{R}^{d'} \to Q \to Q' \to 0$, where $\text{pd}_R Q' < \infty$ and $\dim Q' < \dim \tilde{R}$. Now from the diagram

$$\begin{array}{ccccccccc}
0 & \to & L & \to & \tilde{R}^t \oplus N & \to & Q & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \tilde{R}^{d'} & \to & Q' & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
& & 0 & & 0 \\
\end{array}$$
of exact sequences, we obtain the following short exact sequence

\[ 0 \to L \oplus \tilde{R}^d \to N \oplus \tilde{R}^t \to Q' \to 0 \cdots \]  

(2)

This shows that we can assume \( \text{dim} Q < \text{dim} \tilde{R} \) in (1) and hence, by an easy spectral sequence argument, \( \chi(Q, M) = 0 \). From (1) we obtain \( \chi(M, N) = -\chi(M, \tilde{R}^t) + \chi(M, L) \). On the other hand, since rank of \( L = t + \text{rank}_R N \geq t \), we have the following short exact sequence:

\[ 0 \to \tilde{R}^t \to L \to N' \to 0 \cdots \]  

(3)

where \( \text{dim} N' = \text{dim} N \), depth \( N' \geq \text{dim} N' - 1 \). This implies that \( \chi(M, N') = \chi(M, L) - \chi(M, \tilde{R}^t) = \chi(M, N) \). Hence we are done.

**Remark.** Recall that on a Cohen-Macaulay local ring \( R \) of dimension \( n \), given any pair of modules \( M, N \) such that \( \text{pd}_R M < \infty \) and \( \ell(M \otimes_R N) < \infty \), the following equality holds:

\[ \max\{i \mid \text{Tor}_i(M, N) \neq 0\} = n - \text{depth} M - \text{depth} N. \]

This follows easily by applying “acyclicity Lemma” of Peskine and Szpiro [P–S] to the complex obtained by tensoring a minimal resolutions of \( M \) with \( N \). When \( N' \) is Cohen-Macaulay \( \chi(M, N') = \ell(M \otimes N') > 0 \). So we assume that depth \( N' = \text{dim} N' - 1 \).

**1.5 Definition.** Given a pair of modules \( M, N \) over a Gorenstein ring \( R \) such that \( \text{pd}_R M < \infty, \ell(M \otimes N) < \infty \) and \( \text{dim} M + \text{dim} N = \text{dim} R \), a module \( N' \) will be called a **companion module** for \( N \) if the following hold:

(i) depth \( N' = \text{dim} N' - 1 \),

(ii) \( \text{dim} N' = \text{dim} N \) and
(iii) \( \chi(M, N) = \chi(M, N') \).

We saw in (1.4), given any \( N \), how to construct \( N' \)—a companion module for \( N \) (equations (2) and (3) in (1.4)) by using Auslander-Bridger's theorem. We will use this notion in the proofs of our next Theorem (1.6) and Theorem (1.9).

When depth \( N = \dim N - 1 \), we take \( N \) itself as its companion module.

**1.6 Theorem.** Let \( R \) be Gorenstein and let \( M \) be a perfect module. Write \( \tilde{M} = \Ext^t_R(M, R) \) where \( s = \pd_R M \). Then \( \chi(M, N) \geq 0 \) for any module \( N \) such that \( \ell(M \otimes N) < \infty \) and \( \dim M + \dim N = \dim R \), if for any module \( N' \) such that \( \ell(\tilde{M} \otimes N') < \infty \) and \( \dim \tilde{M} + \dim N' < \dim R \), we have \( \ell(\Tor_1(\tilde{M}, N')) - \ell(\tilde{M} \otimes N') \geq 0 \).

**Proof.** By Proposition (1.4), we can replace \( N \) by a companion module and thus can assume that depth \( N = \dim N - 1 \) and hence \( \chi(M, N) = \ell(M \otimes N) - \ell(\Tor_1(M, N)) \) (Remark 1.4). Let \( \dim M = r \), \( \dim N = s \).

We know that the spectral sequences \( \{\Ext^i(\Tor_j(M, N), R)\}_{i,j=0} \) and \( \{\Ext^i(M, \Ext^j(N, R))\}_{i,j=0} \) converge to the same limit. Since \( \ell(\Tor_j(M, N)) < \infty \), \( \Ext^i(\Tor_j(M, N), R) = 0 \) for \( i \neq n \) and when \( i = n \), it is 0 for \( j > 1 \). Since \( \pd M = s = \dim N \) and depth \( N = s - 1 \), \( \Ext^i(M, \Ext^j(N, R)) = 0 \) for \( i > s \); when \( i \leq s \), it is again zero for \( j < r \), since height \( \ann_R N = r \) and \( R \) is Cohen-Macaulay. Thus the above spectral sequences give rise to the following exact sequence

\[
0 \to \Ext^{s-2}(M, \Ext^{r+1}(N, R)) \to \Ext^s(M, \Ext^r(N, R)) \\
\to \Ext^n(M \otimes N, R) \to \Ext^{s-1}(M, \Ext^{r+1}(N, R)) \to 0. \tag{1}
\]
Denote $\text{Ext}^s(M, R)$ by $\tilde{M}$ and note that $\tilde{M}$ is also a perfect module. Then (1) shows

$$\ell(M \otimes N) = \ell(\text{Ext}^n(M \otimes N, R)) \geq \ell(\text{Ext}^{n-1}(M, \text{Ext}^{r+1}(N, R)))$$

$$= \ell(\text{Tor}_1(\tilde{M}, \text{Ext}^{r+1}(N, R))).$$

On the other hand, $\text{Ext}^n(\text{Tor}_1(M, N), R) \simeq \text{Ext}^{s}(M, \text{Ext}^{r+1}(N, R)).$

This implies $\ell(\text{Tor}_1(M, N)) = \ell(\tilde{M} \otimes \text{Ext}^{r+1}(N, R))$. . . . (2).

We note that $\dim \text{Ext}^{r+1}(N, R) < \dim N$. Hence we obtain our theorem from the inequality below:

$$\chi(M, N) = \ell(M \otimes N) - \ell(\text{Tor}_1(M, N))$$

$$\geq \ell(\text{Tor}_1(\tilde{M}, \text{Ext}^{r+1}(N, R))) - \ell(\tilde{M} \otimes \text{Ext}^{r+1}(N, R))$$

**Corollary (to the proof of the above Theorem)** Let $M$ be as in the above theorem and let $N$ be such that $\ell(M \otimes N) < \infty$ and $\dim M + \dim N = \dim R$. Suppose that $\text{depth } N = \dim N - 1$. Then $\chi(M, N) \geq \ell(\text{Tor}_1(\tilde{M}, \text{Ext}^{r+1}_R(N, R))) - \ell(\tilde{M} \otimes \text{Ext}^{r+1}_R(N, R))$, where $r = \dim M$.

We will use this observation in the proof of Theorem 1.8.

**1.7.** We break the content of our main theorem (Theorem 1.9) into two parts: Proposition 1.7 and Theorem 1.8. Theorem (1.8) covers a very important case—the case when the perfect module $M_r$ has finite length. In several counter-examples, described in the introduction, we come across this situation.

**Proposition.** Let $R$ be a local Gorenstein ring of dimension $n$. Suppose that vanishing fails in $R$ i.e. there exist $R$-modules $M', N'$ such that $\text{pd}_R M' < \infty$, $\ell(M' \otimes N') <$
∞, \dim M' + \dim N' < \dim R and \chi(M', N') \neq 0. Then there exists an R-sequence 
\{y_1, \ldots, y_{n-d}\}, a module M of finite length and finite projective dimension over a local
Gorenstein ring \(\widetilde{R}\) and a prime ideal \(P\) of \(\widetilde{R}\) of height 1 such that

(i) \(\widetilde{R} = R/(y_1, \ldots, y_{n-d}), M = M_r/(y_1, \ldots, y_{n-d})M_r;\) here \(M_r\) is a perfect R-module
and \(\{y_1, \ldots, y_{n-d}\}\) form a maximal \(M_r\)-sequence

(ii) \(\chi^R(M, \widetilde{R}/P) > 0\) and

(iii) \(\chi^R(M, Q) = 0\) if \(\dim Q < \dim \widetilde{R} - 1\).

Proof. We know, by Proposition 1.3 [D1], that \(\chi(M', S) = 0\) for any \(R\)-module \(S\) such
that \(\ell(M' \otimes S) < \infty, \dim M' + \dim S < \dim R\) and \(\dim S \leq 1\). Hence \(\dim N' \geq 2\). Choose
one \(N''\) with least dimension among the set of all modules \(N'\) such that \(\chi(M', N') \neq 0\).
Let \(d - 1 = \dim N''\). Since \(\dim M' + \dim N'' < \dim R\), one can choose an \(R\)-sequence 
\(\{x_1, \ldots, x_d\}\) in \(\text{ann}_R M'\) such that \(\ell(N''/(x_1, \ldots, x_{d-1})N'') < \infty\). Let \(\mathfrak{a}\) denote the ideal
generated by \(x_1, \ldots, x_d\). Then \(\chi(R/\mathfrak{a}, N'') = 0\) (Lemma 1, (1.1)). Now considering short
exact sequences of the type

\[0 \to M_1 \to (R/\mathfrak{a})^{a_0} \to M' \to 0, \quad 0 \to M_2 \to (R/\mathfrak{a})^{a_1} \to M_1 \to 0\]

e tc. for a finite number of times we obtain a perfect module \(M_r\) such that \(\dim M_r =
n - d, \ell(M_r \otimes N'') < \infty\) and \(\chi(M_r, N'') \neq 0\). We can choose \(\{y_1, \ldots, y_{n-d}\}\), an \(R\)-sequence contained in \(\text{ann}_R N''\), such that \(\ell(M_r/(y_1, \ldots, y_{n-d})M_r) < \infty\). Write \(\widetilde{R} = \frac{R}{(y_1, \ldots, y_{n-d})}\) and \(M = M_r/(y_1, \ldots, y_{n-d})M_r\). Then \(\chi^R(M_r, N'') = \chi^R(M, N'') > 0\)
(Lemma 1.4). Moreover, if \(\dim Q < \dim \widetilde{R} - 1(= \dim N'')\), \(\chi^R(M_r, Q) = \chi^R(M, Q)\) (by
Lemma 1.4) and \(\chi^R(M, Q) = 0\) since \(\dim Q < \dim \widetilde{R} - 1 = \dim N''\) and dimension of \(N''\)
is least for the vanishing to fail while intersecting with \( M_r(M') \).

Let \( \{ P_1, \ldots, P_r \} \) be the associated primes of \( N'' \) such that \( \dim \bar{R}/P_i = \dim N'' \). If \( q \in \text{Ass}_R(N'') \), \( q \neq P_i \) for any \( i \), \( 1 \leq i \leq r \), then \( \dim \bar{R}/q < \dim \bar{R} - 1 \) and hence \( \chi^R(M, \bar{R}/q) = 0 \). Thus \( \chi^R(M, N'') = \sum_{i=1}^{r} \ell(N''_i) \chi^R(M, \bar{R}/P_i) \). Since \( \chi^R(M, N'') > 0 \), we must have \( \chi^R(M, \bar{R}/P_i) > 0 \) for at least one \( i \), \( 1 \leq i \leq t \). Hence the Proposition follows.

1.8. Theorem. Let \( R \) be Gorenstein. Let \( M \) be a perfect module of finite length such that \( \chi(M, Q) = 0 \) for any module \( Q \) with \( \dim Q < \dim R - 1 \). Let \( P \) be a prime ideal of height 1 such that \( \chi(M, R/P) \neq 0 \), say \( \chi(M, R/P) > 0 \). Let \( N \) be a companion module for \( P^t \) for \( t \gg 0 \). Assume that \( \chi(M, \text{Ext}_R^1(N, R)) \geq 0 \). Then there exist modules \( M', N' \) such that \( \ell(M') < \infty \), \( M' \) is perfect, \( \dim N' < \dim R \), \( \text{pd}_R N' < \infty \), \( \chi(M', N') = 0 \) and \( \chi_2(M', N') < 0 \).

Proof. By assumption \( \chi(M, R/P^t) = \ell(R_P/P^t R_P) \chi(M, R/P) \). Hence \( \chi(M, R/P) = \lim_{t \to \infty} \chi(M, R/P^t) / \ell(R_P/P^t R_P) \). Consider the exact sequence

\[ 0 \to P^t \to R \to R/P^t \to 0 \]

We have \( \chi(M, R/P^t) = \ell(M) - \chi(M, P^t) \). Since \( \dim R_P = 1 \), it follows that

\[ \lim_{t \to \infty} \chi(M, R/P^t) / \ell(R_P/P^t R_P) = \lim_{t \to \infty} -\chi(M, P^t) / \ell(R_P/P^t R_P) \]

since \( \chi(M, R/P) > 0 \), this implies \( \chi(M, P^t) < 0 \). for \( t \gg 0 \).

Hence \( \chi(M, N) < 0 \) (since \( \chi(M, N) = \chi(M, P^t) \) by Definition 1.5). By Corollary to Theorem (1.6) we obtain

\[ 0 > \chi(M, N) \geq \ell(\text{Tor}_1(M, \text{Ext}_R^1(N, R))) - \ell(M \otimes \text{Ext}_R^1(N, R)), \]

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Hence,
\[ \ell(\text{Tor}_1(M, \text{Ext}^1_R(N, R))) < \ell(M \otimes \text{Ext}^1_R(N, R)) \]  
(1)

If \( \dim \text{Ext}^1_R(N, R) < \dim R - 1 \), then \( \chi(M, \text{Ext}^1_R(N, R)) = 0 \) (by Theorem 1.2 and by assumption). This implies that \( \chi_2(M, \text{Ext}^1_R(N, R)) < 0 \) by (1). If \( \dim \text{Ext}^1_R(N, R) = \dim R - 1 \), let \( x \) be a non-zero-divisor contained in \( \text{ann}_R \text{Ext}^1_R(N, R) \). Write \( \bar{R} = R/xR \).

We can construct an exact sequence

\[ 0 \rightarrow V \rightarrow \bar{R}^{d_1} \rightarrow \cdots \rightarrow \bar{R}^{d_0} \rightarrow \text{Ext}^1_R(N, R) \rightarrow 0 \]

such that \( \chi(M, V) = \chi(M, \text{Ext}^1_R(N, R)) \), \( \chi(M, V) = \chi(M, \text{Ext}^1_R(N, R)) \) and \( V \) is Cohen-Macaulay (note \( \chi(M, \bar{R}) = 0 \) by Lemma 1(1.1)). By Theorem (1.2), \( \chi(M, V) = -\chi(M, \bar{V}) \) where \( \bar{V} = \text{Ext}^1_R(V, R) \). Since, by hypothesis, \( \chi(M, Q) = 0 \) if \( \dim Q < \dim R - 1 \), we have

\[ \chi(M, V) = \sum_{\dim R/q = \dim V} \ell(V_q) \chi(M, R/q) = \sum_{\dim R/q = \dim V} \ell(V_q) \chi(M, R/q) = \chi(M, \bar{V}). \]

(\( V_q \) is the Matlis dual of \( V_q \), hence \( \ell(V_q) = \ell(\bar{V}_q) \)).

Thus \( \chi(M, V) = -\chi(M, V) \). Hence \( \chi(M, \text{Ext}^1_R(N, R)) = -\chi(M, \text{Ext}^1_R(N, R)) \leq 0 \) (by assumption). This again implies that \( \chi_2(M, \text{Ext}^1_R(N, R)) < 0 \) by (1).

Let \( x \) be a non-zero-divisor contained in \( \text{ann}_R \text{Ext}^1_R(N, R) \). Write \( \bar{R} = R/xR \). Then by Theorem 2, (1.3) we obtain an exact sequence of modules over \( \bar{R} \)

\[ 0 \rightarrow \text{Ext}^1_R(N, R) \rightarrow N' \rightarrow L \rightarrow 0 \]

such that \( \text{pd}_R N' < \infty \) and \( L \) is maximal Cohen-Macaulay over \( \bar{R} \).

It is easy to see that \( \chi(M, N') = 0 \) (since \( \ell(\bar{M}) < \infty \) and \( \text{pd}_R N' < \infty \), [D1]). Since \( \dim L = \dim R - 1 \) and \( L \) is Cohen-Macaulay, \( \text{Tor}_i^R(\bar{M}, L) = 0 \) for \( i \geq 2 \). Thus \( \chi_2(\bar{M}, N') = \chi_2(\bar{M}, \text{Ext}^1_R(N, R)) < 0 \). So by taking \( M' = \bar{M} \), we have our theorem.
1.9. **Proof of the main theorem.** Proof of Part (1) follows easily from Proposition (1.7). Proof of Part (2) is identical to the proof of Theorem (1.8), along with Proposition 1.7, due to the following facts: (1) Any $M_r$ sequence is also an $\tilde{M}_r$ sequence, (2) $\tilde{M}_r/(y_1, \ldots, y_{n-d})\tilde{M}_r \simeq (M_r/(y_1, \ldots, y_{n-d})M_r)\nu = (\overline{M}_r)\nu$ and (3) For any $\overline{R}(= R/(y_1, \ldots, y_{n-d}))$ module $N'$, $\chi_i^R(\tilde{M}_r, N') = \chi_i^R((\overline{M}_R)\nu, N')$ by Lemma (1.4). (This $\overline{M}_r$ is our $M$ in Theorem (1.8)).

**Corollary.** $\chi_2$ can be negative on complete intersections of various dimensions even for pairs of modules $M'$, $N'$ s.t. $M'$ is perfect, pd$_R N' < \infty$, dim $M' +$ dim $N' <$ dim $R$ and $\chi(M', N') = 0$.

This follows from the counter example in [D-H-M] and its generalization in [L]. Let $R = k[x, y, u, v]_m/(xy - uv)$, where $m$ is maximal ideal generated by the images of $x$, $y$, $u$ and $v$. We constructed a module $M$ of finite length and finite projective dimension over $R$ such that $\chi(M, R/P) = 1$, where $P = (x, u) \in R$. Consider the exact sequence

$$0 \rightarrow P^t \rightarrow R \rightarrow R/P^t \rightarrow 0$$

Since $P^t = P(t^t)$, dim $P^t = 3$ and depth $P^t = 2$, we can take $P^t$ as its own companion module. Since dim Ext$^1(P^t, R) \leq 1$, $\chi(M, \text{Ext}^1(P^t, R)) = 0$. Thus all the conditions in the hypothesis of the above Theorem are satisfied. Hence the desired result follows.

Similar arguments can be applied to counterexamples in [L] where

$$R = k[x_1, \ldots, x_n, y_1, \ldots, y_n]_m/\Sigma x_i y_i$$

and $P$ is the ideal generated by $(x_1, \ldots, x_n)$. Here one has to take note of the fact that $R/P$ is a regular local ring and hence for any prime ideal $q \supset P$, $\chi^R(M, R/q) = 0$. This
is due to the following facts: (a) \( \chi^R(M, R/q) = \sum (-1)^i \chi^R(\text{Tor}_i^R(M, R/P), R/q) \) where \( \tilde{R} = R/P \), (b) \( \ell(\text{Tor}_i^R(M, R/P)) < \infty, \forall i, 0 \leq i \leq \text{pd}_R M \) and \( \dim R/q < \dim \tilde{R} \), and (c) vanishing holds on the regular local ring \( \tilde{R} \).

1.10. The case of 3-dimensional Gorenstein rings are discussed in the following theorem.

**Theorem.** Let \( R \) be a Gorenstein ring of dimension 3. Suppose that the vanishing does not hold on \( R \). Then there exists modules \( M' \) and \( N' \) such that both \( M' \) and \( N' \) have finite projective dimension over \( R \), \( \ell(M' \otimes N') < \infty \), \( \dim M' + \dim N' < 3 \), \( \chi(M', N') = 0 \) and \( \chi_2(M', N') < 0 \).

**Proof.** Since \( R \) is of dimension 3, if the vanishing fails for \( R \), then it fails for a pair of modules \( M, N \) such that \( \ell(M) < \infty \), \( \text{pd}_R M < \infty \) and \( N = R/P \) where \( P \) is a prime ideal with \( \dim R/P = 2 \). Recall that for any local Groenstein ring \( R \) and for any \( R \)-module \( M \) of finite projective dimension if there exists a module \( N \) such that \( \dim N \leq 1 \), \( \dim M + \dim N < \dim R \) and \( \ell(M \otimes_R N) < \infty \), then \( \chi(M, N) = 0 \) (Proposition 1.3, [D1]). Hence we have \( \chi(M, R/P^{(t)}) = \chi(M, R/P^t) \). Without any loss of generality, can assume \( \chi(M, R/P) > 0 \) (changing \( M \) to \( \tilde{M} \) if needed!). Consider the exact sequence

\[
0 \to P^{(t)} \to R \to R/P^{(t)} \to 0.
\]

Since \( \chi(M, R/P) > 0 \), we have (as in the proof of Theorem (1.9))

\[
\chi(M, P^{(t)}) < 0
\]

Since depth \( P^{(t)} = 2 \), \( \dim P^{(t)} = 3 \), \( P^{(t)} \) can be taken as its companion module. Moreover, since \( \text{Ext}^1(P^{(t)}, R) \simeq \text{Ext}^2(R/P^{(t)}, R) \) and \( R \) is Gorenstein of dimension 3, we have
\[ \dim \text{Ext}^1(P^{(t)}, R) \leq 1. \] Hence \( \chi(M, \text{Ext}^1(P^{(t)}, R)) = 0 \) [D1]. Thus, by Theorem (1.8), we are done.

Corollary. Let \( R = k[x, y, z, w]/f(x, y, z, w) \), \( m = (x, y, z, w) \). Suppose that \( f \) is a homogeneous cubic polynomial for which Proj(R) is regular and \( k \) is algebraically closed. Roberts and Srinivas have shown that there exist modules of finite length and finite projective dimension with negative intersection multiplicities over \( R \) [R–S]. Our theorem above establishes the negativity of \( \chi_2 \) on such rings.

**Section 2**

In this section we concentrate on ramified regular local rings. The \( \chi_i \)-conjecture remains to be settled on such rings. However we have an inter-relation between \( \chi_2 \) and \( \chi \) which we pointed out in [D2]. For the sake of completeness of this paper we sketch a proof here.

**2.1 Theorem.** Let \( R \) be regular local. Then positivity holds when one of the intersecting modules is Cohen-Macaulay, if \( \chi_2 \) is valid over \( R \) for pairs of modules \( M', N' \) such that \( \ell(M' \otimes N') < \infty \) and \( \dim M' + \dim N' < \dim R \).

**Proof.** Let \( M \) be perfect, \( N \) be another module such that \( \ell(M \otimes N) < \infty \) and \( \dim M + \dim N = \dim R \). By Proposition 1.4 we can assume depth \( N = \dim N - 1 \). Then Corollary to Theorem 1.6 shows that

\[
\chi(M, N) \geq \ell(M, \text{Ext}^{r+1}_R(N, R)) = \ell(M \otimes \text{Ext}^{r+1}_R(N, R))
\]

Thus if \( \chi(M, \text{Ext}^{r+1}_R(N, R)) > 0 \), we are done.
Suppose that $\chi_2(\tilde{M}, \text{Ext}^{r+1}(N, R)) = 0$. By the $\chi_2$-conjecture we know that this implies $\text{Tor}_2(\tilde{M}, \text{Ext}^{r+1}(N, R)) = 0$. Then it follows (from (1) and (2) in the proof of Theorem (1.6)) that $\ell(M \otimes N) = \ell(\text{Tor}_1(\tilde{M}, \text{Ext}^{r+1}(N, R)) + \ell(\tilde{M} \otimes \text{Ext}^r(N, R))$ and $\ell(\text{Tor}_1^R(M, N)) = \ell(\tilde{M} \otimes \text{Ext}^{r+1}(N, R))$.

Hence $\chi(M, N) = \ell(\tilde{M} \otimes \text{Ext}^r(N, R)) + \chi_2(\tilde{M}, \text{Ext}^r(N, R)) = \ell(\tilde{M} \otimes \text{Ext}^r(N, R)) > 0$.

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