ON MODULES OF FINITE PROJECTIVE DIMENSION OVER COMPLETE INTERSECTIONS

S. P. Dutta

Abstract. Recently Avramov and Miller proved that over a local complete intersection ring
$(R, m, k)$ in characteristic $p > 0$, a finitely generated module $M$ has finite projective dimension
if for some $i > 0$ and for some $n > 0$, $\text{Tor}_i^R(M, f^n_R) = 0$ being the Frobenius map repeated
$n$ times. They used the notion of “complexity” and several related theorems. Here we offer
a very simple proof of the above theorem without using “complexity” at all.

Recently Avramov and Miller [A-M] gave an important characterization for modules
of finite projective dimension over complete intersections in positive characteristic. To
describe their result we need the following set-up:

Let $(R, m)$ be a local ring of characteristic $p > 0$ with residue field $k = R/m$. Let
$f : R \to R, f(x) = x^p$, denote the Frobenius map and let $f^n$ denote the map $f$ repeated
$n$-times. We denote by $f^nR$ the bi-$R$-algebra $R$, having the structure of an $R$-algebra from
the left by $f^n$ and from the right by identity. For any $R$-module $M, F^n_R(M)$ will stand
for $M \otimes_R f^nR$ and $f^nN$ will denote the module $N$ viewed as an $R$-module via $f^n[P - S]$.
Avramov and Miller proved the following:

Theorem [A–M]. Let $M$ be a finitely generated module over a local complete intersection
ring $(R, m, k)$. If for some $i > 0$ and for some $n > 0$, $\text{Tor}_i^R(M, f^n_R) = 0$, then $M$ is of

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finite projective dimension.

Their proof used the notion of “complexity” and several related theorems. We here intend to provide a much simpler proof which does not use “complexity” at all. The only well-known theorems we exploit are the following: the theorem on flatness of the frobenius due to Kunz [K, 3.3] and Herzog’s theorem on characterization of finite projective dimension [H, 3.1].

Historically work on such problems started in 1969 with Kunz’s theorem [K, 3.3] on equivalence of flatness of $f^n$ for all $n \geq 1$ with the regularity of the local ring $R$. Next Peskine and Szpiro established the following [P–S, 1.7]: If a finitely generated $R$-module $M$ has finite projective dimension over $R$, $\text{Tor}_i^R(M, f^n_R) = 0$ for all $i, n \geq 1$. Then Herzog ([H], 3.1) proved the converse: If $M$ is finitely generated and $\text{Tor}_i^R(M, f^n_R) = 0$ for all $i > 1$ and infinitely many $n$, $M$ has finite projective dimension over $R$. The theorem of Avramov and Miller stated above is a much desired extension of Herzog’s theorem to complete intersections. For modules $M$ of finite length over a complete interection this author showed [D, 1.9] that $\ell(F^n_R(M)) \geq p^{nd}\ell(M)$ where $d$ denotes dimension of $R$ and sign of equlity holds when $M$ is of finite projective dimension; in [M, 1.1] Miller proved the converse and deduced the following: If for some $i > 0$, $\lim_{n \to \infty} \ell(\text{Tor}_i^R(M, f^n_R))/p^{nd} = 0$, then $M$ has finite projective dimension.

We accomplish our proof in the following steps.

**Step 0** Without any loss of generality we can assume that $R$ is complete and $R = S/\underline{x}$ where $S$ is a complete regular local ring of characteristic $p > 0$ and $\underline{x} = (x_1 \ldots, x_r)$ is the ideal generated by an $S$-sequence $x_1, \ldots, x_r$. Let $d$ be dimension of $R$ (henceforth dim);
then $r + d = \dim S$. We know by Kunz’s Theorem ([K], 3.3) that $f^n : S \to S : f^n(x) = x^{p^n}$ is a flat map $\forall n > 0$.

**Step 1** Since $S \overset{f^n} \longrightarrow S$ is flat, $S/\mathfrak{a} \overset{f^n} \longrightarrow S/\mathfrak{a}^{p^n}$ is flat (base change). Hence $R \overset{f^n} \longrightarrow R$ can be factored as

$$R = S/\mathfrak{a} \overset{f^n} \longrightarrow S/\mathfrak{a}^{p^n} \overset{\eta_n} \longrightarrow R = S/\mathfrak{a}$$

where $\eta_n$ is the natural surjection.

Thus $f^n R = \eta_n \cdot \tilde{f}^n$.

Let $M$ be a finitely generated $R$-module. Consider an exact sequence (a presentation of $M$)

$$R^{t_1} \overset{\phi} \longrightarrow R^{t_0} \longrightarrow M \longrightarrow 0. \quad (1)$$

Apply $\otimes R f^n S$ and obtain an exact sequence

$$(S/\mathfrak{a}^{p^n})^{t_1} \overset{\phi \otimes f^n} \longrightarrow (S/\mathfrak{a}^{p^n})^{t_0} \longrightarrow F^n_S(M) \longrightarrow 0 \quad (2)$$

(1) and (2) imply that

$$M \otimes R \tilde{f}^n (S/\mathfrak{a}^{p^n}) \simeq F^n_S(M). \quad (3)$$

Since $\tilde{f}^n$ is flat, we have

$$\text{Tor}_i^R(M, f^n R) = \text{Tor}_i^R (F^n_S(M), S/\mathfrak{a})$$

(4)

where $R_n = S/\mathfrak{a}^{p^n}$

**Step 2** We want to show that $\text{Tor}_1^R(M, f^n R) = 0$ implies that $\text{Tor}_i^R(M, f^n R) = 0$ for $i \geq 1$. It is enough to show that $\text{Tor}_2^R(M, f^n R) = 0$. By flatness of $\tilde{f}^n$ we will be done by
showing:

\[ \text{Tor}_1^{R_n}(F^n_S(M), S/\mathfrak{a}) = 0 \quad \text{implies that} \quad \text{Tor}_2^{R_n}(F^n_S(M), S/\mathfrak{a}) = 0. \]

We know \( S/\mathfrak{a}^n \) has a filtration such that successive quotients are isomorphic to \( S/\mathfrak{a} \). We have the following exact sequences:

\[
0 \to K_1 \to S/\mathfrak{a}^n \to S/\mathfrak{a} \to 0
\]

\[
0 \to K_2 \to K_1 \to S/\mathfrak{a} \to 0
\]

\[
0 \to K_{t_n} \to K_{t_{n-1}} \to S/\mathfrak{a} \to 0
\]

where \( K_{t_n} = S/\mathfrak{a} \). Since \( \text{Tor}_1^{R_n}(F^n_S(M), S/\mathfrak{a}) = 0 \), we obtain by going up along the above exact sequences successively that \( \text{Tor}_1^{R_n}(F^n_S(M), K_1) = 0 \). This implies that \( \text{Tor}_2^{R_n}(F^n_S(M), S/\mathfrak{a}) = 0 \). Hence \( \text{Tor}_2^{R}(M, f^nR) = 0 \).

**Step 3** We want to show that \( \text{Tor}_1^{R}(M, f^nR) = 0 \) implies that \( \text{Tor}_1^{R}(M, f^{n+1}R) = 0 \). Hence we need to show that \( \text{Tor}_1^{R_{n+1}}(F^n_{S}(M), S/\mathfrak{a}) = 0 \), provided \( \text{Tor}_1^{R_{n}}(F^n_{S}(M), S/\mathfrak{a}) = 0 \). Recall from Step 2 that since \( \text{Tor}_1^{R_{n}}(F^n_{S}(M), S/\mathfrak{a}) = 0, \text{Tor}_1^{R_{i}}(F^n_{S}(M), S/\mathfrak{a}) = 0 \) for \( i \geq 1 \) and from Step 1 that \( R_n \to R_{n+1} \) is flat. Hence (tensoring the above equation by \( fR_{n+1} \))

\[ \text{Tor}_1^{R_{n+1}}(F^n_{S}(M), S/\mathfrak{a}^n) = 0 \quad \text{for} \quad i \geq 1 \] (5)

Consider the short exact sequences

\[ 0 \to S/(x_1, x_2, \ldots, x_r) \to S/(x_1^{p+1}, x_2^p, \ldots, x_n^p) \to S/\mathfrak{a}^p \to 0 \] (6)

and

\[ 0 \to S/\mathfrak{a}^p \to S/(x_1^{p+1}, x_2^p, \ldots, x_r^p) \to S/(x_1, x_2, \ldots, x_r) \to 0 \] (7)
Applying (5), from (6) we obtain

\[ \text{Tor}_i^{R_{n+1}}(F_{S_1}^{n+1}(M), S/(x_1^p, \ldots, x_r^p)) \]

\[ \simeq \text{Tor}_i^{R_{n+1}}(F_{S_1}^{n+1}(M), S/(x_1^{p+1}, x_2^p, \ldots, x_r^p)) \quad \text{for } i \geq 1 \]  
(8)

and applying (5), from (7) we obtain

\[ \text{Tor}_i^{R_{n+1}}(F_{S_1}^{n+1}(M), S/(x_1^{p+1}, x_2^p, \ldots, x_r^p)) \]

\[ \hookrightarrow \text{Tor}_i^{R_{n+1}}(F_{S_1}^{n+1}(M), S/(x_1^p, x_2^p, \ldots, x_r^p)) \quad \text{for } i \geq 1 \]  
(9)

which is an isomorphism for \( i \geq 2 \) and an injection for \( i = 1 \).

Note that the composition of the two maps (8) and (9) is the one induced by multiplication by \( x_1^p \).

Combining (6), (7), (8) and (9) we observe that

\[ S/(x_1, x_2^p, \ldots, x_r^p) \xrightarrow{x_1^p} S/(x_1^p, x_2^p, \ldots, x_r^p) \]

is the 0-map and it induces an injection between

\[ \text{Tor}_i^{R_{n+1}}(F_{S_1}^{n+1}(M), S/(x_1^p, x_2^p, \ldots, x_r^p)) \hookrightarrow \text{Tor}_i^{R_{n+1}}(F_{S_1}^{n+1}(M), S/(x_1^p, x_2^p, \ldots, x_r^p)) \]

for \( i \geq 1 \).

Thus \( \text{Tor}_i^{R_{n+1}}(F_{S_1}^{n+1}(M), S/(x_1^p, x_2^p, \ldots, x_r^p)) = 0 \). Repeating this process \( (r - 1) \) times we obtain

\[ \text{Tor}_i^{R_{n+1}}(F_{S_1}^{n+1}(M), S/(x^p)) = 0 \quad \text{for } i \geq 1. \]

**Step 4** Step 1, Step 2, Step 3 and the well-known theorem due to Herzog ([H] 3.1) proves the assertion stated in the theorem at the beginning.
Remark. With notations as above, over a complete intersection \( R = S/x \), for a finitely generated module \( M \), we have \( \text{Pd}_R M < \infty \) if and only if \( \text{Pd}_R F^g(M) < \infty \).

The proof follows easily from Step 1.

References


Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, IL 61801
U.S.A.
é-mail: dutta@math.uiuc.edu