Semmes spaces

JANG-MEI WU∗

To understand the underlying structure of a metric space, one seeks a parametrization of a special type. For example, every Riemannian manifold homeomorphic to the 2-sphere is conformally equivalent to $S^2$.

In his 1996 Revista Matemática Iberoamericana papers [12, 13], Stephen Semmes gave unexpected counterexamples to several natural conjectures on the bilipschitz and quasisymmetric parametrizations of metric $n$-spheres. His examples are geometrically self-similar manifolds modeled on the decomposition spaces associated with the Whitehead continuum, Bing’s dogbone, or Bing’s double; these spaces admit metrics that are smooth Riemannian outside a totally disconnected closed set and, in some sense, indistinguishable from the standard metric on $S^3$ geometrically and measure theoretically, and yet are not quasisymmetrically equivalent to $S^3$.

Through these examples, he addresses the roles of wildness, shrinkability and linking in questions of parametrization, and expresses his philosophical views on mappings and spaces. The usefulness of Semmes’ construction is not limited to the problems at hand. Work on Semmes-type spaces there-

∗wu@math.uiuc.edu.
after by Heinonen and Rickman [5] and by Pankka and Rajala [8] further highlights the topological properties of the spaces that admit or receive geometrically controlled branched covering maps.

In these papers of Semmes, ideas linking analysis and topology unroll in a story-telling style—more like a novel than a textbook. One can relax and watch Semmes effortlessly connecting the dots and unfolding the facts. In the end, the reader is rewarded with a mystery solved.

We now give a sample of Semmes’ ideas and their implications.

**Semmes metrics.** Despite considerable attention in recent years, the problem of characterizing metric $n$-spheres that are bilipschitz or quasisymmetrically equivalent to the standard $S^n$ is still far from understood. There exist finite 5-dimensional polyhedra (double suspension of homology 3-spheres with nontrivial fundamental groups) homeomorphic to the standard sphere $S^5$ but not bilipschitz equivalent to $S^5$—an observation of Siebenmann and Sullivan (1979) based on deep work of Cannon (1978) and Edwards (1980). It is unknown whether these polyhedra are quasisymmetrically equivalent to $S^5$.

When $\mathbb{R}^n$ is equipped with a path metric $D_\omega(x,y)$ associated with a continuous strong $A_\infty$ weight $\omega$ ($\omega dx$ a doubling measure and $D_\omega(x,y)^n$ comparable to the $\omega$-measure, $\int_{B_{x,y}} \omega$, of the smallest Euclidean ball $B_{x,y}$ containing $x$ and $y$), the geometry of the space $(\mathbb{R}^n, D_\omega)$ is in many ways indistinguishable from $\mathbb{R}^n$ (David and Semmes 1990, Semmes 1993). For example, the metric $D_\omega$ is quasi-equivalent to the Euclidean metric in the sense that every $D_\omega$-ball $B$ contains a Euclidean ball and is contained in a Euclidean ball of comparable radii (in general very different from the radius of $B$); the $\omega$-measure of any $D_\omega$-ball of radius $r$ is comparable to $r^n$ (Ahlfors $n$-regular); and every $D_\omega$-ball contains a definite portion in $\omega$-measure that is uniformly bilipschitz equivalent to a subset of $\mathbb{R}^n$ in the Euclidean metric. Moreover $(\mathbb{R}^n, D_\omega)$ supports Sobolev and weak $(1,1)$-Poincaré inequalities which are crucial for differential calculus. Must $(\mathbb{R}^n, D_\omega)$ be bilipschitz equivalent to $\mathbb{R}^n$?

In [13] Semmes found a strong $A_\infty$ weight $\omega$ so that the associated space $(\mathbb{R}^3, D_\omega)$ is not bilipschitz equivalent to $\mathbb{R}^3$.

Semmes’ idea is to create a metric in $\mathbb{R}^3$ in terms of the distance function to a geometrically nice but wild Cantor set. Under Semmes’ metric this Cantor set has a small Hausdorff dimension. Precisely, let $\mathbf{N}$ be a geometrically self-similar Antoine’s necklace constructed in such a way that all tori used are similar and all tori in the same generation are congruent as illustrated in [10, p. 73]. The complement $\mathbb{R}^3 \setminus \mathbf{N}$ is non-simply connected. Then, for a fixed $s > 0$,

$$\omega(x) = \min(1, \text{dist}(x, \mathbf{N})^s)$$
is a strong $A_\infty$ weight in $\mathbb{R}^3$. The Hausdorff dimension of the necklace $N$ in $(\mathbb{R}^n, D_\omega)$ is at most $(1 + s/6)^{-1}$. On the other hand, every homeomorphism $h$ of $\mathbb{R}^3$ maps $N$ to a set with non-simply connected complement; this implies that the Hausdorff dimension of $h(N)$ is at least 1 for the Euclidean metric.

The spaces $(\mathbb{R}^3, D_\omega)$ and $\mathbb{R}^3$ therefore are not bilipschitz equivalent.

Jacobians of quasiconformal mappings are classical $A_\infty$-weights. The problem of characterizing weights in $\mathbb{R}^n$ that are comparable to a quasiconformal Jacobian is related to the bilipschitz parametrization problem. Semmes’ example can be rephrased to give a counterexample the conjecture that every strong $A_\infty$-weight is a quasiconformal Jacobian.

The topological obstructions above are restricted to dimension 3 or higher. Good metrics on 2-spheres that do not admit bilipschitz parametrization by $S^2$ have been constructed later by Laakso (2002) and Bishop (2007).

The idea of constructing metrics based on distance is versatile. It can be adapted to create new metrics on subsets of $\mathbb{R}^3$ exploiting their topological characteristics in such a way that the obstruction to a particular problem caused by the Euclidean metric disappears and the solvability is determined by the topological nature of the sets. We call any metric constructed with this goal a Semmes-type metric.

**Non-Euclidean Picard Theorem.** Non-constant quasiregular mappings (higher dimension analogues of analytic functions or multivalent analogues of quasiconformal maps) from $\mathbb{R}^n$ to $\mathbb{R}^n$ can omit only finitely many values; and for any finite set of points in $\mathbb{R}^3$ there exists a quasiregular mapping from $\mathbb{R}^3$ into $\mathbb{R}^3$ omitting exactly those points – striking theorems of Rickman in 1980 and 1985.

Equipped with a Semmes-type metric, subsets of $S^3$ become more amenable to receiving quasiregular maps. A sharp non-Euclidean Picard-type theorem in dimension 3 of Pankka and Rajala [8] inspired by Semmes’ construction says that if $L$ is either an unknot (flat circle) or a Hopf link (two flat circles linked once) in $S^3$, then there exists a Riemannian metric $g$ in $S^3 \setminus L$ so that $(S^3 \setminus L, g)$ receives non-constant quasiregular maps from $\mathbb{R}^3$; i.e., is quasiregularly elliptic; on the other hand, if $L$ is a link in $\mathbb{R}^3$ and there exists a Riemannian metric $g$ in $S^3 \setminus L$ so that $(S^3 \setminus L, g)$ is quasiregularly elliptic, then $L$ must be an unknot or a Hopf link.

In the case of the classical Picard theorem, the non-existence of (non-constant) analytic functions into a twice punctured plane is due to the fact that the fundamental group $\pi_1(\mathbb{C} \setminus \{0, 1\})$ is a free group on two generators. The same topological obstruction occurs in the non-Euclidean theorem: the fundamental group of $\pi_1(S^3 \setminus L)$ contains a free group of rank 2 as a subgroup if $L$ is any link except the unknot or the Hopf link.
It is unknown however whether $\mathbb{S}^3 \setminus W_h$, the complement of a Whitehead continuum, can be equipped with a Semmes-type metric so that it is quasiregularly elliptic [8].

**Semmes spaces.** When is a metric $n$-sphere $(X,d)$ quasisymmetrically equivalent to $\mathbb{S}^n$? A complete characterization is known only for dimensions 1 and 2 ([14], [2]). Conditions of Semmes [11] and of Bonk and Kleiner [2] imply that *if a metric 2-sphere is linearly locally contractible (every ball of radius $r$ is contractible in the ball of radius $Cr$ with the same center) and Ahlfors 2-regular (there exists a measure $\mu$ on the space so that the $\mu$-measure of every ball of radius $r$ is comparable to $r^2$ uniformly) then it is quasisymmetrically equivalent to $\mathbb{S}^2$.*

Could a metric $n$-sphere which resembles $\mathbb{S}^n$ geometrically (linearly locally contractible) and measure-theoretically (Ahlfors $n$-regular) fail to be quasisymmetrically equivalent to $\mathbb{S}^n$?

*Semmes’ negative example in dimension 3 is a geometrically self-similar metric space $(\mathbb{R}^3/\text{Bd},d)$ modeled on the decomposition of $\mathbb{R}^3$ with respect to Bing’s double [12].*

The classical construction of R.H. Bing in geometric topology gives an involution in $\mathbb{S}^3$ whose fixed point set is a double horned sphere. Bing’s double is a set constructed following Bing’s procedure topologically, not necessarily geometrically. One construction of Bing’s double $\text{Bd}$ starts with a solid smooth torus $T$ standardly embedded in $\mathbb{R}^3$ and two smooth tori $T_1$ and $T_2$ linked and embedded in the interior of $T$ as illustrated in [1, Fig. 3, p. 357], or in [3, Fig 9-1, p. 63]. Let $\phi_j: T \to T_j$, $j = 1,2$ be diffeomorphisms, $S_1 = \{1, 2\}^l$, $\alpha = (\alpha_1, \ldots, \alpha_l) \in S_l$ and $\phi_\alpha = \phi_{\alpha_1} \circ \cdots \circ \phi_{\alpha_l} \circ \phi_{\alpha_l}$. Bing’s double is

$$\text{Bd} = \bigcap_{l=1}^{\infty} \bigcup_{\alpha \in S_l} \phi_\alpha T.$$  

The complement $\mathbb{R}^3 \setminus \text{Bd}$ is not simply connected, and as a topological space $\mathbb{R}^3/\text{Bd}$ is $\mathbb{R}^3$ for nontrivial reasons.

Semmes’ idea is to embed $\mathbb{R}^3/\text{Bd}$ into $\mathbb{R}^4$ by unknotted and resizing the tori in the construction geometrically. As a first step, $\mathbb{R}^3 \setminus T$ is embedded in $\mathbb{R}^3 \times \{0\}$ by inclusion, and the linked tori $T_1 \cup T_2$ are mapped diffeomorphically onto tori, similar to $T$ and of size $\lambda$ times that of $T$, which are contained in two mutually disjoint Euclidean balls in $(\text{int} T) \times \{0\}$. The embedding is then extended to a diffeomorphism $\theta$ from $\mathbb{R}^3$ into $\mathbb{R}^4$ by an unknotted argument. Careful construction of $\theta$ allows $\theta(T_1)$ and $\theta(T_2)$ to assume the role of $T$ and the unknotted and resizing procedure to be iterated geometrically. At the limit we obtain a map that descends to a homeomor-
phism from $\mathbb{R}^3/Bd$ into $\mathbb{R}^4$. Semmes’ geometrical realization of $\mathbb{R}^3/Bd$ is a 3-dimensional submanifold of $\mathbb{R}^4$ smooth outside a Cantor set $Bd^*$. The space $(\mathbb{R}^3/Bd, d_\lambda)$ with the metric induced by the ambient Euclidean metric in $\mathbb{R}^4$ through the embedding is quasiconvex, linearly locally contractible and Ahlfors 3-regular and smooth except for well-controlled degeneracies near $Bd^*$. Moreover it satisfies the Sobolev and Poincaré inequalities needed for analysis.

However the Semmes space $(\mathbb{R}^3/Bd, d_\lambda)$ is not quasisymmetrically equivalent to $\mathbb{R}^3$.

Semmes’ elegant explanation of this fact goes as follows. Suppose $h$ is a homeomorphism from $\mathbb{R}^3/Bd$ onto $\mathbb{R}^3$. All $l$-th generation tori in $\mathbb{R}^3/Bd$ are similar to $T$ and have diameter $\lambda^l$. Their images in $\mathbb{R}^3$ circulate around $h(\theta(T))$ at least $2^l$ times. Therefore at least one of $2^l$ image tori, call it $\tau_l$, must have a longitude of length at least $c_0 > 0$, for every $l \geq 1$. Since $\text{diam} \tau_l \to 0$ as $l \to \infty$, the tori $\tau_l$ can not be uniformly well-shaped, therefore $h$ can not be quasisymmetric. This heuristic argument can be made precise by a lemma of Freedman and Skora (1987) using relative homology.

Any number of linked tori may be used in the first step of defining Bing’s double. In case one torus $T_1$ is self-linked in $T$ in such a way that a meridian of $T$ and a longitude of $T_1$ form a Whitehead link as in [3, Fig. 9-7, p. 68] or in [10, p. 72], the resulting intersection is called a Whitehead continuum $Wh$. In case $k$ ($\geq 3$) tori $\bigcup_{1 \leq j \leq k} T_j$ are linked in $T$ as in [3, Fig. 9-9, p. 71] or in [10, p. 73], the resulting set is an Antoine’s necklace.

Semmes’ geometrization extends to all decomposition spaces of $\mathbb{R}^3$ defined by an initial package with a topological self-similarity. With an additional contractibility condition, the resulting spaces are generalized manifolds possessing all metric properties mentioned above for the space $(\mathbb{R}^3/Bd, d_\lambda)$. We call these spaces and other non-self similar ones constructed in this spirit Semmes-type spaces.

**Branched covering maps.** It seems that the existence of a bilipschitz parametrization for metric spheres is a rarity and that a concrete geometrical characterization is difficult. Heinonen and Rickman [5] however showed that all spaces constructed in a geometrically self-similar manner from initial packages of Semmes on $S^3$ with an additional contractibility condition, admit BLD-maps (maps of bounded length distortion –multivalent analogues of bilipschitz maps) onto $S^3$.

The example arising from the Bing’s double decomposition space leads to a space $(S^3/Bd, d_\lambda)$ that is homeomorphic to $S^3$, although quasisymmetrically inequivalent to $S^3$, but can be mapped onto $S^3$ by a BLD-map whose branch set contains a wild Cantor set. This particular example shows a
sharp contrast between the finite-to-one and injective cases and the power of Semmes-type metrics.

It follows from the above that there exists a branched cover $F: S^3 \rightarrow S^3$ (discrete open map) so that for no homeomorphism $h: S^3 \rightarrow S^3$ is $F \circ h: S^3 \rightarrow S^3$ quasiregular. An important question remains open: whether every such branch covering map $F$ is topologically conjugate to a quasiregular map, i.e., there exist homeomorphisms $g, h: S^3 \rightarrow S^3$ so that $g \circ F \circ h$ is quasiregular [5].

**Quasisymmetric parametrization.** At a meeting in 2005, Juha Heinonen suggested that we work on the question of quasisymmetric parametrization of the double suspension of homology 3-spheres $\Sigma^2 H^3$ [6, Question 12]. With no idea whether the answer would be yes or no, we set out in Fall 2006 to read Edwards' explicit construction of a homeomorphism between $S^5$ and a particular $\Sigma^2 H^3$ (work of 1980, arXiv 2006). Our hope was that Edwards’ map could be modified to be quasisymmetric; this task turned out to be more ambitious than originally expected. On the other hand, there is a subtle connection between the double suspension problem and the decomposition theory at the topological level [3, p.103].

As we struggled to make progress in quasisymmetric parametrization, experimenting with Semmes-type spaces built from classical examples in decomposition theory was fascinating. With Heinonen [7] we showed that the natural conditions mentioned earlier for good parametrization are also insufficient in dimension 4 or higher. More specifically, the decomposition space $\mathbb{R}^3/Wh$ associated with the Whitehead continuum admits a Semmes-type metric that is linearly locally contractible and Ahlfors 3-regular but $(\mathbb{R}^3/Wh) \times \mathbb{R}^m$ is not quasisymmetrically equivalent to $\mathbb{R}^{3+m}$, for any $m \geq 1$.

The complement of the Whitehead continuum $Wh$ in $S^3$ is a contractible non-compact 3-manifold that is not homeomorphic to $\mathbb{R}^3$. The decomposition space $\mathbb{R}^3/Wh$ is not $\mathbb{R}^3$, but $(\mathbb{R}^3/Wh) \times \mathbb{R}$ is homeomorphic to $\mathbb{R}^4$. The nonexistence of the quasisymmetric parametrization is due to the different roles of the meridians in the homotopy and the homology in the Whitehead construction and their roles in estimating moduli of surface families.

$\mathbb{R}^3/Wh$ is only one case of a non-trivial manifold factor of $\mathbb{R}^4$. By a theorem of Edwards and Miller [4], cell-like closed 0-dimensional upper semicontinuous decomposition spaces $\mathbb{R}^3/G$ are manifold factors of $\mathbb{R}^4$, $\mathbb{R}^3/G \times \mathbb{R} \approx \mathbb{R}^4$. Decomposition spaces satisfying Edwards and Miller’s conditions are definable by defining sequences consisting of unions of cubes-with-handles (handlebodies), see Lambert and Sher (1968) and Sher and Alford (1968). This class provides a natural environment for testing quasisymmetric parametrization.
With Pankka [9] we consider a subclass of decomposition spaces $\mathbb{R}^3/G$ that are manifold factors and admit defining sequences $(X_k)$ consisting of handlebodies of controlled topological complexity. As self-similar spaces these spaces may be equipped with Semmes-type metrics with controlled geometry that are linearly locally contractible and Ahlfors 3-regular.

We have noted that the existence of a quasisymmetric parametrization of $\mathbb{R}^3/G \times \mathbb{R}^m$ by $\mathbb{R}^{3+m}$ for any $m \geq 0$ imposed a necessary constraint on the geometry (growth of the handlebodies and the scaling factor of the metric) in terms of the topology (genus, welding and circulation of the handlebodies), which is needed for the quasi-invariance of the modulus. On the other hand, a strong self-similar welding structure on the decomposition suffices to guarantee the existence of a quasisymmetric parametrization of $(\mathbb{R}^3/G, d)$ for a properly chosen Semmes-type metric. Here, the growth defines how fast the handlebodies propagate; the welding describes an embedding relation between handlebodies of two consecutive generations; and the circulation, in some sense, sums up the (unsigned) winding numbers of handlebodies of one particular generation inside the previous one. Even for this subclass the gap between the known necessary and the sufficient conditions remains wide.

Semmes spaces combine a new kind of metrization with classical topology in a subtle and mysterious manner. In the field of quasiconformal analysis, if one searches, Semmes-type spaces exist everywhere.

References


