Ternary forms with lots of zeros

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Conference/Workshop on Inverse Moment Problems, IMS
National University of Singapore, Dec. 19, 2013
Outline of the talk

- Statement of the problem and the conjectures (joint with Greg Blekherman)
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- Relation to cones
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- Arxiv paper 2007
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- Arxiv paper 2007
- New material (joint with Greg Blekherman and ??)
In this table, we see, in order: $2r$, $r^2$, $\frac{(2r-1)(2r-2)}{2}$, their maximum, and then $\frac{3}{2}r(r - 1) + 1$. The smaller of the last two is the upper bound on $|\mathcal{Z}(p)|$ for $p \in P_{3,2r}$.
Let $I(n, d) = \{(i_1, \ldots, i_n) : 0 \leq i_k \in \mathbb{Z}, \sum i_k = d\}$ and for $i \in I(n, d)$ write the multinomial coefficient $c(i) = \frac{d!}{i_1! \cdots i_n!}$. Write $x^i = x_1^{i_1} \cdots x_n^{i_n}$ and for a real form $p$ of degree $d$ in $n$ variables, scale the coefficients by:

$$p = \sum_{i \in I(n, d)} c(i) a(p; i) x^i$$

Then the \textit{Fischer inner product} is defined by

$$[p, q] = \sum_{i \in I(n, d)} c(i) a(p; i) a(q; i).$$

For $\alpha \in \mathbb{R}^n$ define $(\alpha \cdot)^d = (\alpha \cdot x)^d$, then it is easy to see that

$$[p, (\alpha \cdot)^d] = p(\alpha).$$
It is not so hard to see from this that $P_{n,2r}$ and $Q_{n,2r}$ are dual cones. The dual cone to $\Sigma_{n,2r}$ is easily understood. If $p \in \Sigma^*_{n,2r}$, then $[p, h^2] \geq 0$ for all $h$ of degree $r$. Write a general form of degree $r$ with the coefficients as variables

$$h(x) = \sum_{\ell \in I(n,r)} t(\ell) x^\ell \implies [p, h^2] = \sum_{\ell} \sum_{\ell'} a(p; \ell + \ell') t(\ell) t(\ell') := H_p(t)$$

This construction was done by Sylvester for ternary quartics; he called it the “catalecticant”. It’s important to note that

$$H_{(\alpha \cdot)^{2r}}(t) = \left( \sum_{\ell} \alpha^\ell t(\ell) \right)^2$$

and so the rank of $H_p$ is a lower bound on the width of $p$. When $P_{n,2r} = \Sigma_{n,2r}$, the duals are equal and the two are the same. In general, this doesn’t happen for the other cases.
Suppose

\[ p(x, y, z) = \sum_{i \in I(3, d)} \frac{d!}{i!j!k!} a(p; (i, j, k)) x^i y^j z^k. \]

Then

\[ p(x, y, z) = \sum_{t=1}^{N} (a_t x + b_t y + c_t z)^d \iff a(p; (i, j, k)) = \sum_{t=1}^{N} a_t^i b_t^j c_t^k \]

If \( c_t \neq 0 \), this can be rewritten as

\[ a(p; (i, j, k)) = \sum_{t=1}^{N} a_t^i b_t^j c_t^k \]

or

\[ a(p; (i, j, k)) = \int X^i Y^j d\mu, \] where \( \mu \) has \( N \) masses of weight \( c_t \) at \( (a_t c_t, b_t c_t) \).
Suppose
\[ p(x, y, z) = \sum_{i \in I(3,d)} d! \frac{a(p; (i, j, k)) x^i y^j z^k}{i! j! k!}. \]

Then
\[ p(x, y, z) = \sum_{t=1}^{N} (a_t x + b_t y + c_t z)^d \iff a(p; (i, j, k)) = \sum_{t=1}^{N} a_t^i b_t^j c_t^k \]

If \( c_t \neq 0 \), this can be rewritten as
\[ a(p; (i, j, k)) = \sum_{t=1}^{N} c_t^d \left( \frac{a_t}{c_t} \right)^i \left( \frac{b_t}{c_t} \right)^j \]
Suppose

\[ p(x, y, z) = \sum_{i \in I(3,d)} \frac{d!}{i!j!k!} \cdot a(p; (i, j, k))x^iy^jz^k. \]

Then

\[ p(x, y, z) = \sum_{t=1}^{N} (a_t x + b_t y + c_t z)^d \iff a(p; (i, j, k)) = \sum_{t=1}^{N} a_t^i b_t^j c_t^k \]

If \( c_t \neq 0 \), this can be rewritten as

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or \( a(p; (i, j, k)) = \int X^i Y^j d\mu \),

where \( \mu \) has \( N \) masses of weight \( c_t^d \) at \( \left( \frac{a_t}{c_t}, \frac{b_t}{c_t} \right) \).
ContourPlot[
  \((x^2 - 1)^2 (x^2 - 9)^2 + (y^2 - 1)^2 (y^2 - 9)^2 \) == 10, \{x, -4, 4\}, \{y, -4, 4\}]

This is not a beamer talk.
\(\text{Out}[751]= \quad 6075 x^8 - 3105 x^4 y^4 + 46 \sqrt{345} x^2 y^6 - \\
121500 x^6 z^2 + 12420 x^4 y^2 z^2 + 31050 x^2 y^4 z^2 - \\
368 \sqrt{345} x^2 y^4 z^2 + 3402 y^6 z^2 + 716850 x^4 z^4 - \\
124200 x^2 y^2 z^4 + 736 \sqrt{345} x^2 y^2 z^4 - 55161 y^4 z^4 - \\
1093500 x^2 z^6 + 166212 y^2 z^6 + 492075 z^8\)

\(\text{In}[752]:= \quad \text{Factor}[\%751 \/. \{z \rightarrow 0\}, \text{Extension} \rightarrow \text{Sqrt}[345]]\)

\(\text{Out}[752]= \\
\frac{1}{23} \sqrt{\frac{15}{23}} x^2 \left(3 \sqrt{345} x^2 - 23 y^2\right)^2 \left(3 \sqrt{345} x^2 + 46 y^2\right)\)
In[756]= ContourPlot[(%751 /. z -> 1) == 10000, {x, -4, 4}, {y, -4, 4}]

Out[756]=
Out[748] = 25 x^8 + 72 x^6 y^2 + 144 x^5 y^3 + 194 x^4 y^4 + 144 x^3 y^5 + 
72 x^2 y^6 + 25 y^8 - 572 x^6 z^2 - 144 x^5 y z^2 - 1436 x^4 y^2 z^2 - 
1728 x^3 y^3 z^2 - 1436 x^2 y^4 z^2 - 144 x y^5 z^2 - 572 y^6 z^2 + 
4192 x^4 z^4 + 1584 x^3 y z^4 + 6584 x^2 y^2 z^4 + 1584 x y^3 z^4 + 
4192 y^4 z^4 - 9720 x^2 z^6 - 1440 x y z^6 - 9720 y^2 z^6 + 8100 z^8
In[750]= `ContourPlot[(%748 /. z -> 1) == 100, {x, -4, 4}, {y, -4, 4}]`

Out[750]=

-4 -2 0 2 4
-4
-2
0
2
4
Thank you for your patience.